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The Equilibrium Configuration of a Two Phase Medium with a Density Change

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1. INTRODUCTION

In this paper we use some well-known methods of the Calculus of Variations to prove the existence of a solution to an elliptic boundary-value problem with an internal free boundary arising in the equilibrium configuration of a two-phase medium having a density difference between the phases. In Section 2 we formulate the problem and define the notation to be used. In Section 3 we solve the problem, using two simple lemmas which are proved in Section 4.

2. NOTATION AND STATEMENT OF THE PROBLEM

We are given a material which can be found in a solid phase, denoted by I, and a liquid phase, denoted by II. We denote by ρ_i, K_i $(i = 1, 2)$ the density and conductivity of the material in phase I and phase II, respectively; K_i may be a function of the temperature T in each phase. Suppose that the phase transition from I to II occurs at the temperature T_c.

We will denote by E_h the domain

$$0 < x < 1, \qquad 0 < y < h$$

of the x, y plane. Let B_1 be the base of E_h:

$$B_1 : 0 < x < 1, \qquad y = 0,$$

and let B_2 denote the remaining sides and upper boundary of E_h:

$$B_2 : \begin{cases} x = 1, & 0 < y < h, \\ x = 0, & 0 < y < h, \\ x = h, & 0 < x < 1. \end{cases}$$

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We define Γ_h to be the entire boundary of E_h:

$$\Gamma_h = B_1 \cup B_2 .$$

Let $T_1 > T_c$, and let $f(x)$, $0 < x < 1$ be a smooth function on B_1 which at least on some subinterval I of B_1 assumes values which are less that T_c:

$$f(x) < T_c \text{ on } I.$$

For the sake of simplicity we will assume that I is a single subinterval of B_1. We will denote by g the function defined on Γ_h by

$$g = T_1 \text{ on } B_2 , \qquad g = f \text{ on } B_1 .$$

The problem with which we will be concerned is the following:

Problem 1

Let $M > 0$ be a given number. Prove that there exists a number h, a Jordan arc C in E_h joining the endpoints of I, and a function $T(x, y)$ in E_h, such that

(A) T obeys the boundary and interface conditions:

$$T = g \quad \text{on } \Gamma_h \tag{1.1}$$

and

$$T = T_c \text{ on } C; \tag{1.2}$$

(B) C divides E_h into two subdomains E_h^1, E_h^2, for which

$$(K_i(T)T_x)_x + (K_i(T)T_y)_y = 0 \text{ in } E_h^i, \qquad i = 1, 2, \tag{1.3}$$

$$T < T_c \text{ in } E_h^1, \qquad T > T_c \text{ in } E_h^2, \tag{1.4}$$

and

$$K_1(T_c) \operatorname{grad} T^{(1)} = K_2(T_c) \operatorname{grad} T^{(2)} \text{ on } C, \tag{1.5}$$

where superscripts (1) and (2) denote the limiting values on C from within E_h^1 and E_h^2, respectively, and the quantities $K_1(T_c)$ and $K_2(T_c)$ are the conductivities, respectively, of the phase I and phase II materials at the critical temperature T_c;

(C) If A_h^1, A_h^2 denote the areas of E_h^1, E_h^2, respectively, then

$$\rho_1 A_h^1 + \rho_2 A_h^2 = M. \tag{1.6}$$

THEOREM. *For any M there exists a solution to Problem* 1, *comprising the value h, the curve C and the function $T(x, y)$.*

Physically, Problem 1 corresponds to the partial freezing of a mass M of liquid in a trough whose walls are maintained, together with the air temperature, at T_1. Due to the change in density the actual volume occupied by the material in the steady state must be determined as part of the solution.

3. Proof of the Theorem

We will prove the theorem using the method of [2] to remove condition Eq. (1.5) and then applying some known results of the calculus of variations to the resulting problem.

By defining the function

$$K(T) = \begin{cases} K_1(T), & T \leqslant T_c \\ K_2(T), & T > T_c \end{cases}$$

and using the Kirchhoff transformation

$$U(x, y) = \int_{T_c}^{T(x,y)} K(\zeta)\, d\zeta,$$

we may reformulate Problem 1 as follows:

Problem 2

Find a number h, a curve C and a function $U(x, y)$ for which

$$U = \int_{T_c}^{g} K(\zeta)\, d\zeta \qquad \text{on} \quad \Gamma_h \tag{2.1}$$

and

$$U = 0 \qquad \text{on} \quad C, \tag{2.2}$$

where C and the domains E_h^1, E_h^2 together with their areas A_h^1, A_h^2 are defined as in Problem 1;

$$\Delta U = 0 \text{ throughout } E_h \tag{2.3}$$

$$U < 0 \text{ in } E_h^1; \qquad U > 0 \text{ in } E_h^2 \tag{2.4}$$

and Eq. (1.6) holds.

It is clear that for a given M there are infinitely many ways of defining a smooth or even piecewise smooth (but not necessarily harmonic) function U, a curve C, and a value h so that Eq. (1.6) is valid. However, if we ignore Eq. (1.6), then there is a uniquely determined function U for each value of h, satisfying Eqs. (2.1)–(2.4) which, due to the smoothness of f, can be characterized as the solution to the variational problem of minimizing the Dirichlet integral

$$D_h[U] = \iint_{E_h} (U_x^2 + U_y^2)\, dx\, dy$$

among all piecewise smooth functions U satisfying Eq. (2.1). In fact, the following result is well-known [1]: Given any value h, there is a unique function $U^h(x, y)$ satisfying Eq. (2.1), which is harmonic in the domain E_h and minimizes the integral $D_h[U]$ among all piecewise smooth functions U satisfying Eq. (2.1). This function defines a curve C_h along which U^h vanishes, and which lies completely within E_h.

It is clear that U^h need not satisfy Eq. (1.6). To prove the existence theorem we examine the values

$$M_h = \rho_1 A_h^1 + \rho_2 A_h^2$$

with the aid of known properties of the values

$$d_h = D_h[U^h]$$

for varying h. Let us define

$$y^h = \max y : (x, y) \in C_h ;$$

i.e., y^h is the maximum height of all points of C_h. Then the following lemmas which are proved in Section 4 are valid.

LEMMA 1. *For $h_1 < h_2$,*

$$d_{h_1} > d_{h_2} .$$

LEMMA 2. *For all h sufficiently large,*

$$y^h \leqslant Y$$

with Y independent of h.

We note (see [1]) that U^h and hence C_h vary continuously with h. However, this implies that M_h varies continuously with h, and since

$$h \max(\rho_1, \rho_2) \geqslant M_h \geqslant h \min(\rho_1, \rho_2),$$

we have

$$M_h \to \infty \qquad \text{as} \quad h \to \infty,$$

which by Lemma 2 implies the existence of some h for which

$$M_h = M.$$

Thus a solution to the problem exists.

4. Proof of Lemmas 1 and 2

Proof of Lemma 1. Let $h_1 < h_2$. Then U^{h_2} solves the minimum problem for the Dirichlet integral on the domain E_{h_2} which contains E_{h_1}. Extending the function U^{h_1} to E_{h_2} by having it assume the constant value T_1 on $E_{h_2} - E_{h_1}$ results in a continuous piecewise smooth function U^* on E_{h_2} for which

$$D_{h_1}[U^{h_1}] = D_{h_2}[U^*] > D_{h_2}[U^{h_2}],$$

thus proving Lemma 1.

Lemma 2 may be proved by using the maximum principle. An interesting alternative method can be based on the use of certain well-known properties of the Dirichlet integral, as follows:

Proof of Lemma 2. Assume that as $h \to \infty$, $y_h \to \infty$. Let $h_n \to \infty$ be a sequence of values, and assume that $y_{h_n} \to \infty$. Let $\delta > 0$ be a fixed value, and for each y_{h_n} let us partially subdivide the domain $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant y_{h_n}$ into $[y_{h_n}/\delta]$ strips. Since by Lemma 1, for all $n = 1, 2,\dots$,

$$D_{h_n}[U^{h_n}] \leqslant D_{h_1}[U^{h_1}],$$

for each n there is some strip S_n on which

$$D_{S_n}[U^{h_n}] < \epsilon_n ,$$

where $\epsilon_n \to 0$ as $n \to \infty$. However, this is impossible, according to a well-known estimate of the total variation of a function in terms of its Dirichlet integral (see [1, Lemma 1.4a]), which asserts the existence of a horizontal line in S_n, on which

$$\max | U(P_2) - U(P_1)| \leqslant (\epsilon_n/\delta)^{1/2},$$

with the maximum taken over all pairs P_1, P_2 of points on this line. In fact, by the definition of S_n, the curve C_{h_n} crosses this strip, and the horizontal line in S_n contains points at which $U = T_1$ and $U = T_c$, whence

$$| T_1 - T_c | \leqslant (\epsilon_n/\delta)^{1/2}.$$

But since $\epsilon_n \to 0$ for $n \to \infty$, this inequality is impossible and the lemma is proved.

References

1. R. Courant, "Dirichlet's Principle, Conformal Mapping and Minimal Surfaces," Interscience Publishers, Inc., New York, 1950.
2. A. Solomon, A steady state phase change problem, *Math. Comp.* 21 (1967), 355–359.